数学ノート

目 次

1　３°刻みの三角比

1.1　３°刻みの三角比

$$\begin{cases} \sin 0\,^\circ = 0 \\ \cos 0\,^\circ = 1 \end{cases}$$

$$\begin{cases} \sin 3\,^\circ = \dfrac{\sqrt{2}(\sqrt{3}+1)(\sqrt{5}-1)-2(\sqrt{3}-1)\sqrt{\sqrt{5}+5}}{16} \\ \cos 3\,^\circ = \dfrac{\sqrt{2}(\sqrt{3}-1)(\sqrt{5}-1)+2(\sqrt{3}+1)\sqrt{\sqrt{5}+5}}{16} \end{cases}$$

$$\begin{cases} \sin 6\,^\circ = \dfrac{\sqrt{2}\sqrt{3}(\sqrt{5}-1)\sqrt{\sqrt{5}+5}-2(\sqrt{5}+1)}{16} \\ \cos 6\,^\circ = \dfrac{\sqrt{2}(\sqrt{5}-1)\sqrt{\sqrt{5}+5}+2\sqrt{3}(\sqrt{5}+1)}{16} \end{cases}$$

$$\begin{cases} \sin 9\,^\circ = \dfrac{\sqrt{2}(\sqrt{5}+1)-(\sqrt{5}-1)\sqrt{\sqrt{5}+5}}{8} \\ \cos 9\,^\circ = \dfrac{\sqrt{2}(\sqrt{5}+1)+(\sqrt{5}-1)\sqrt{\sqrt{5}+5}}{8} \end{cases}$$

$$\begin{cases} \sin 12\,^\circ = \dfrac{\sqrt{2}\sqrt{\sqrt{5}+5}-\sqrt{3}(\sqrt{5}-1)}{8} \\ \cos 12\,^\circ = \dfrac{\sqrt{2}\sqrt{3}\sqrt{\sqrt{5}+5}+(\sqrt{5}-1)}{8} \end{cases}$$

$$\begin{cases} \sin 15\,^\circ = \dfrac{\sqrt{2}(\sqrt{3}-1)}{4} \\ \cos 15\,^\circ = \dfrac{\sqrt{2}(\sqrt{3}+1)}{4} \end{cases}$$

$$\begin{cases} \sin 18\,^\circ = \dfrac{\sqrt{5}-1}{4} \\ \cos 18\,^\circ = \dfrac{\sqrt{2}\sqrt{\sqrt{5}+5}}{4} \end{cases}$$

$$\begin{cases} \sin 21\,^\circ = \dfrac{(\sqrt{3}+1)(\sqrt{5}-1)\sqrt{\sqrt{5}+5}-\sqrt{2}(\sqrt{3}-1)(\sqrt{5}+1)}{16} \\ \cos 21\,^\circ = \dfrac{(\sqrt{3}-1)(\sqrt{5}-1)\sqrt{\sqrt{5}+5}+\sqrt{2}(\sqrt{3}+1)(\sqrt{5}+1)}{16} \end{cases}$$

$$\begin{cases} \sin 24\,^\circ = \dfrac{2\sqrt{3}(\sqrt{5}+1)-\sqrt{2}(\sqrt{5}-1)\sqrt{\sqrt{5}+5}}{16} \\ \cos 24\,^\circ = \dfrac{2(\sqrt{5}+1)+\sqrt{2}\sqrt{3}(\sqrt{5}-1)\sqrt{\sqrt{5}+5}}{16} \end{cases}$$

$$\begin{cases} \sin 27\,^\circ = \dfrac{2\sqrt{\sqrt{5}+5}-\sqrt{2}(\sqrt{5}-1)}{8} \\ \cos 27\,^\circ = \dfrac{2\sqrt{\sqrt{5}+5}+\sqrt{2}(\sqrt{5}-1)}{8} \end{cases}$$

$$\begin{cases} \sin 30^\circ = \dfrac{1}{2} \\ \cos 30^\circ = \dfrac{\sqrt{3}}{2} \end{cases}$$

$$\begin{cases} \sin 33^\circ = \dfrac{2(\sqrt{3}-1)\sqrt{\sqrt{5}+5}+\sqrt{2}(\sqrt{3}+1)(\sqrt{5}-1)}{16} \\ \cos 33^\circ = \dfrac{2(\sqrt{3}+1)\sqrt{\sqrt{5}+5}-\sqrt{2}(\sqrt{3}-1)(\sqrt{5}-1)}{16} \end{cases}$$

$$\begin{cases} \sin 36^\circ = \dfrac{\sqrt{2}(\sqrt{5}-1)\sqrt{\sqrt{5}+5}}{8} \\ \cos 36^\circ = \dfrac{\sqrt{5}+1}{4} \end{cases}$$

$$\begin{cases} \sin 39^\circ = \dfrac{\sqrt{2}(\sqrt{3}+1)(\sqrt{5}+1)-(\sqrt{3}-1)(\sqrt{5}-1)\sqrt{\sqrt{5}+5}}{16} \\ \cos 39^\circ = \dfrac{\sqrt{2}(\sqrt{3}-1)(\sqrt{5}+1)+(\sqrt{3}+1)(\sqrt{5}-1)\sqrt{\sqrt{5}+5}}{16} \end{cases}$$

$$\begin{cases} \sin 42^\circ = \dfrac{\sqrt{2}\sqrt{3}\sqrt{\sqrt{5}+5}-(\sqrt{5}-1)}{8} \\ \cos 42^\circ = \dfrac{\sqrt{2}\sqrt{\sqrt{5}+5}+\sqrt{3}(\sqrt{5}-1)}{8} \end{cases}$$

$$\begin{cases} \sin 45^\circ = \dfrac{\sqrt{2}}{2} \\ \cos 45^\circ = \dfrac{\sqrt{2}}{2} \end{cases}$$

1.2 求め方

角度	求め方１	求め方２	求め方３
0 °	○		
3 °			18 °-15 °
6 °			36 °-30 °
9 °			45 °-36 °
12 °			30 °-18 °
15 °		45 °-30 °, 半角	
18 °		５倍角	
21 °			30 °-9 °
24 °			60 °-36 °
27 °			45 °-18 °
30 °	○		
33 °			45 °-12 °
36 °		５倍角	
39 °			45 °-6 °
42 °			60 °-18 °
45 °	○		

1.3 $\cos 18^\circ$ を求める

$$\cos 18^\circ = \frac{\sqrt{2}\sqrt{\sqrt{5}+5}}{4}$$

（証明）

$\theta = 18^\circ$のとき　$5\theta = 90^\circ$

$\cos 5\theta = \cos 90^\circ = 0$　に注目する。

$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$　より、$x = \cos\theta$とおけば、

$16x^5 - 20x^3 + 5x = 0$

$x \neq 0$ より

$16x^4 - 20x^2 + 5 = 0$

$\therefore \quad x^2 = \dfrac{10 \pm \sqrt{100-80}}{16} = \dfrac{10 \pm \sqrt{20}}{16} = \dfrac{10 \pm 2\sqrt{5}}{16} = \dfrac{5 \pm \sqrt{5}}{8}$

$\cos 18^\circ > \cos 30^\circ$より $x > \dfrac{\sqrt{3}}{2}$

$\therefore x^2 > \dfrac{3}{4} = \dfrac{6}{8}$

$\therefore x^2 = \dfrac{5+\sqrt{5}}{8}$

$\therefore x = \dfrac{\sqrt{5+\sqrt{5}}}{\sqrt{8}} = \dfrac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}} = \dfrac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}$

ゆえに、$\cos 18^\circ = \dfrac{\sqrt{2}\sqrt{\sqrt{5}+5}}{4}$

2 三角関数の公式

2.1 n 乗の公式

$$\begin{cases} \sin^2\theta = 1 - \cos^2\theta \\ \cos^2\theta = 1 - \sin^2\theta \end{cases}$$
$$\begin{cases} \sin^4\theta = 1 - 2\cos^2\theta + \cos^4\theta \\ \cos^4\theta = 1 - 2\sin^2\theta + \sin^4\theta \end{cases}$$
$$\begin{cases} \sin^6\theta = 1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta \\ \cos^6\theta = 1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta \end{cases}$$
$$\begin{cases} \sin^8\theta = 1 - 4\cos^2\theta + 6\cos^4\theta - 4\cos^6\theta + \cos^8\theta \\ \cos^8\theta = 1 - 4\sin^2\theta + 6\sin^4\theta - 4\sin^6\theta + \sin^8\theta \end{cases}$$
$$\begin{cases} \sin^{10}\theta = 1 - 5\cos^2\theta + 10\cos^4\theta - 10\cos^6\theta + 5\cos^8\theta - \cos^{10}\theta \\ \cos^{10}\theta = 1 - 5\sin^2\theta + 10\sin^4\theta - 10\sin^6\theta + 5\sin^8\theta - \sin^{10}\theta \end{cases}$$

2.2 倍角の公式

$$\begin{cases} \sin 2\theta = 2\sin\theta\cos\theta \\ \cos 2\theta = 2\cos^2\theta - 1 \end{cases}$$
$$\begin{cases} \sin 3\theta = -4\sin^3\theta + 3\sin\theta \\ \cos 3\theta = 4\cos^3\theta - 3\cos\theta \end{cases}$$
$$\begin{cases} \sin 4\theta = 4\sin\theta\cos\theta(1 - 2\sin^2\theta) \\ \cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1 \end{cases}$$
$$\begin{cases} \sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta \\ \cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta \end{cases}$$
$$\begin{cases} \sin 6\theta = 2\sin\theta\cos\theta(16\sin^4\theta - 16\sin^2\theta + 3) \\ \cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 \end{cases}$$
$$\begin{cases} \sin 7\theta = -64\sin^7\theta + 112\sin^5\theta - 56\sin^3\theta + 7\sin\theta \\ \cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta \end{cases}$$
$$\begin{cases} \sin 8\theta = 8\sin\theta\cos\theta(-16\sin^6\theta + 24\sin^4\theta - 10\sin^2\theta + 1) \\ \cos 8\theta = 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1 \end{cases}$$

2.3 n 倍角の公式

$$\begin{cases} \sin n\theta = \dfrac{(\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n}{2i} \\ \cos n\theta = \dfrac{(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n}{2} \end{cases}$$
$$\begin{cases} \cos n\theta = {}_nC_0\cos^n\theta - {}_nC_2\cos^{n-2}\theta\sin^2\theta + {}_nC_4\cos^{n-4}\theta\sin^4\theta - \cdots \\ \sin n\theta = {}_nC_1\cos^{n-1}\theta\sin\theta - {}_nC_3\cos^{n-3}\theta\sin^3\theta + {}_nC_5\cos^{n-5}\theta\sin^5\theta - \cdots \end{cases}$$

3　公式の証明

3.1　$\sin^n\theta$

(1)$\sin^2\theta = 1-\cos^2\theta$
（証明）
$\sin^2\theta+\cos^2\theta=1$　より
$\sin^2\theta = 1-\cos^2\theta$

(2)$\sin^4\theta = 1-2\cos^2\theta+\cos^4\theta$
（証明）
$\sin^2\theta = 1-\cos^2\theta$　より
$\sin^4\theta = (1-\cos^2\theta)^2$
$= 1-2\cos^2\theta+\cos^4\theta$

(3)$\sin^6\theta = 1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta$
（証明）
$\sin^2\theta = 1-\cos^2\theta$　より
$\sin^6\theta = (1-\cos^2\theta)^3$
$= 1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta$

(4)$\sin^8\theta = 1-4\cos^2\theta+6\cos^4\theta-4\cos^6\theta+\cos^8\theta$
（証明）
$\sin^2\theta = 1-\cos^2\theta$　より
$\sin^8\theta = (1-\cos^2\theta)^4$
$= 1-4\cos^2\theta+6\cos^4\theta-4\cos^6\theta+\cos^8\theta$

(5)$\sin^{10}\theta = 1-5\cos^2\theta+10\cos^4\theta-10\cos^6\theta+5\cos^8\theta-\cos^{10}\theta$
（証明）
$\sin^2\theta = 1-\cos^2\theta$　より
$\sin^{10}\theta = (1-\cos^2\theta)^5$
$= 1-5\cos^2\theta+10\cos^4\theta-10\cos^6\theta+5\cos^8\theta-\cos^{10}\theta$

3.2　$\cos^n\theta$

(1)$\cos^2\theta = 1-\sin^2\theta$

（証明）
$\sin^2\theta+\cos^2\theta=1$　より
$\cos^2\theta = 1-\sin^2\theta$

(2)$\cos^4\theta = 1-2\sin^2\theta+\sin^4\theta$
（証明）

$\cos^2\theta = 1 - \sin^2\theta$　より
$\cos^4\theta = (1 - \sin^2\theta)^2$
$= 1 - 2\sin^2\theta + \sin^4\theta$

(3)$\cos^6\theta = 1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta$
（証明）
$\cos^2\theta = 1 - \sin^2\theta$　より
$\cos^6\theta = (1 - \sin^2\theta)^3$
$= 1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta$

(4)$\cos^8\theta = 1 - 4\sin^2\theta + 6\sin^4\theta - 4\sin^6\theta + \sin^8\theta$
（証明）
$\cos^2\theta = 1 - \sin^2\theta$　より
$\cos^8\theta = (1 - \sin^2\theta)^4$
$= 1 - 4\sin^2\theta + 6\sin^4\theta - 4\sin^6\theta + \sin^8\theta$

(5)$\cos^{10}\theta = 1 - 5\sin^2\theta + 10\sin^4\theta - 10\sin^6\theta + 5\sin^8\theta - \sin^{10}\theta$
（証明）
$\cos^2\theta = 1 - \sin^2\theta$　より
$\cos^{10}\theta = (1 - \sin^2\theta)^5$
$= 1 - 5\sin^2\theta + 10\sin^4\theta - 10\sin^6\theta + 5\sin^8\theta - \sin^{10}\theta$

3.3　２倍角の公式

(1)$\sin 2\theta = 2\sin\theta\cos\theta$
（証明）
$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$　より
$\theta = \alpha = \beta$とおいて、
$\sin 2\theta = 2\sin\theta\cos\theta$

(2)$\cos 2\theta = \cos^2\theta - \sin^2\theta$
（証明）
$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$　より
$\theta = \alpha = \beta$とおいて、
$\cos 2\theta = \cos^2\theta - \sin^2\theta$
また、$\sin^2\theta = 1 - \cos^2\theta$より $\cos 2\theta = 2\cos^2\theta - 1$
また、$\cos^2\theta = 1 - \sin^2\theta$より $\cos 2\theta = 1 - 2\sin^2\theta$

3.4　３倍角の公式

(1)$\sin 3\theta = -4\sin^3\theta + 3\sin\theta$
（証明）

$\sin 3\theta = \sin(2\theta + \theta)$
$= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$
$= 2\sin\theta\cos^2\theta + (1 - 2\sin^2\theta)\sin\theta$
$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta$
$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$
$= -4\sin^3\theta + 3\sin\theta$

(2)$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
（証明）
$\cos 3\theta = \cos(2\theta + \theta)$
$= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$
$= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta$
$= 2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta)$
$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$
$= 4\cos^3\theta - 3\cos\theta$

3.5　４倍角の公式

(1)$\sin 4\theta = 4\sin\theta\cos\theta(1 - 2\sin^2\theta) = 4\sin\theta\cos\theta(2\cos^2\theta - 1)$
（証明）
$\sin 4\theta = 2\sin 2\theta\cos 2\theta$
$= 4\sin\theta\cos\theta(1 - 2\sin^2\theta)$
$= 4\sin\theta\cos\theta(2\cos^2\theta - 1)$

(2)$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1 = 1 - 8\sin^2\theta + 8\sin^4\theta$
（証明）
$\cos 4\theta = 2\cos^2 2\theta - 1$
$= 2(2\cos^2\theta - 1)^2 - 1$
$= 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1$
$= 8\cos^4\theta - 8\cos^2\theta + 1$
また、
$\cos 4\theta = 1 - 2\sin^2 2\theta$
$= 1 - 8\sin^2\theta\cos^2\theta$
$= 1 - 8\sin^2\theta(1 - \sin^2\theta)$
$= 1 - 8\sin^2\theta + 8\sin^4\theta$

3.6　５倍角の公式

(1)$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$
（証明）
$\sin 5\theta = \sin(4\theta + \theta)$
$= \sin 4\theta\cos\theta + \cos 4\theta\sin\theta$

$= 4\sin\theta\cos^2\theta(1-2\sin^2\theta)+(8\sin^4\theta-8\sin^2\theta+1)\sin\theta$
$= 4\sin\theta(1-\sin^2\theta)(1-2\sin^2\theta)+8\sin^5\theta-8\sin^3\theta+\sin\theta$
$= 4\sin\theta(1-3\sin^2\theta+2\sin^4\theta)+8\sin^5\theta-8\sin^3\theta+\sin\theta$
$= 4\sin\theta-12\sin^3\theta+8\sin^5\theta++8\sin^5\theta-8\sin^3\theta+\sin\theta$
$= 16\sin^5\theta-20\sin^3\theta+5\sin\theta$

(2)$\cos 5\theta = 16\cos^5\theta-20\cos^3\theta+5\cos\theta$
（証明）
$\cos 5\theta = \cos(4\theta+\theta)$
$= \cos 4\theta\cos\theta-\sin 4\theta\sin\theta$
$= (8\cos^4\theta-8\cos^2\theta+1)\cos\theta-4\sin^2\theta\cos\theta(2\cos^2\theta-1)$
$= 8\cos^5\theta-8\cos^3\theta+\cos\theta+4\cos\theta(1-\cos^2\theta)(1-2\cos^2\theta)$
$= 8\cos^5\theta-8\cos^3\theta+\cos\theta+4\cos\theta(1-3\cos^2\theta+2\cos^4\theta)$
$= 8\cos^5\theta-8\cos^3\theta+\cos\theta+4\cos\theta-12\cos^3\theta+8\cos^5\theta$
$= 16\cos^5\theta-20\cos^3\theta+5\cos\theta$

3.7　６倍角の公式

(1)$\sin 6\theta = 2\sin\theta\cos\theta(16\sin^4\theta-16\sin^2\theta+3)$
（証明）
$\sin 6\theta = 2\sin 3\theta\cos 3\theta$
$= 2(-4\sin^3\theta+3\sin\theta)(4\cos^3\theta-3\cos\theta)$
$= 2\sin\theta\cos\theta(4\sin^2\theta+3)(4\cos^2\theta-3)$
$= 2\sin\theta\cos\theta(4\sin^2\theta+3)(4-4\sin^2\theta-3)$
$= 2\sin\theta\cos\theta(4\sin^2\theta+3)(-4\sin^2\theta+1)$
$= 2\sin\theta\cos\theta(16\sin^4\theta-16\sin^2\theta+3)$

(2)$\sin 6\theta = 2\sin\theta\cos\theta(16\cos^4\theta-16\cos^2\theta+3)$
（証明）
$\sin 6\theta = 2\sin\theta\cos\theta(16\sin^4\theta-16\sin^2\theta+3)$
$= 2\sin\theta\cos\theta(16(1-2\cos^2\theta+\cos^4\theta)-16(1-\cos^2\theta)+3)$
$= 2\sin\theta\cos\theta(16\cos^4\theta-16\cos^2\theta+3)$

(3)$\cos 6\theta = 32\cos^6\theta-48\cos^4\theta+18\cos^2\theta-1$
（証明）
$\cos 6\theta = 2\cos^2 3\theta-1$
$= 2(4\cos^3\theta-3\cos\theta)^2-1$
$= 2(16\cos^6\theta-24\cos^4\theta+9\cos^2\theta)-1$
$= 32\cos^6\theta-48\cos^4\theta+18\cos^2\theta-1$

(4)$\cos 6\theta = -32\sin^6\theta+48\sin^4\theta-18\sin^2\theta+1$
（証明）
$\cos 6\theta = 32\cos^6\theta-48\cos^4\theta+18\cos^2\theta-1$

$= 32(1-3\sin^2\theta+3\sin^4\theta-\sin^6\theta)-48(1-2\sin^2\theta+\sin^4\theta)+18(1-\sin^2\theta)-1$
$= 32-96\sin^2\theta+96\sin^4\theta-32\sin^6\theta-48+96\sin^2\theta-48\sin^4\theta+18-18\sin^2\theta-1$
$= -32\sin^6\theta+48\sin^4\theta-18\sin^2\theta+1$

3.8　8倍角の公式

(1)$\sin 8\theta = 8\sin\theta\cos\theta(-16\sin^6\theta+24\sin^4\theta-10\sin^2\theta+1)$
（証明）
$\sin 8\theta = 2\sin 4\theta\cos 4\theta$
$= 4\sin 2\theta\cos 2\theta(1-8\sin^2\theta+8\sin^4\theta)$
$= 8\sin\theta\cos\theta(1-2\sin^2\theta)(1-8\sin^2\theta+8\sin^4\theta)$
$= 8\sin\theta\cos\theta(1-8\sin^2\theta+8\sin^4\theta-2\sin^2\theta+16\sin^4\theta-16\sin^6\theta)$
$= 8\sin\theta\cos\theta(-16\sin^6\theta+24\sin^4\theta-10\sin^2\theta+1)$

(2)$\cos 8\theta = 128\cos^8\theta-256\cos^6\theta+160\cos^4\theta-32\cos^2\theta+1$
（証明）
$\cos 8\theta = 2\cos^2 4\theta-1$
$= 2(8\cos^4\theta-8\cos^2\theta+1)^2-1$
$= 2(64\cos^8\theta+64\cos^4\theta+1-128\cos^6\theta-16\cos^2\theta+16\cos^4\theta)-1$
$= 2(64\cos^8\theta-128\cos^6\theta+80\cos^4\theta-16\cos^4\theta)-1$
$= 128\cos^8\theta-256\cos^6\theta+160\cos^4\theta-32\cos^2\theta+1$

3.9　n倍角の公式

$$\begin{cases} \sin n\theta = \dfrac{(\cos\theta+i\sin\theta)^n-(\cos\theta-i\sin\theta)^n}{2i} \\ \cos n\theta = \dfrac{(\cos\theta+i\sin\theta)^n+(\cos\theta-i\sin\theta)^n}{2} \end{cases}$$

（証明）

$\begin{cases} (\cos\theta+i\sin\theta)^n = \cos n\theta+i\sin n\theta \\ (\cos\theta-i\sin\theta)^n = \cos n\theta-i\sin n\theta \end{cases}$ より、

$$\cos n\theta = \frac{(\cos\theta+i\sin\theta)^n+(\cos\theta-i\sin\theta)^n}{2}$$
$$\sin n\theta = \frac{(\cos\theta+i\sin\theta)^n-(\cos\theta-i\sin\theta)^n}{2i}$$

3.10　n倍角の公式

$$\begin{cases} \cos n\theta = {}_nC_0\cos^n\theta-{}_nC_2\cos^{n-2}\theta\sin^2\theta+{}_nC_4\cos^{n-4}\theta\sin^4\theta-\cdots \\ \sin n\theta = {}_nC_1\cos^{n-1}\theta\sin\theta-{}_nC_3\cos^{n-3}\theta\sin^3\theta+{}_nC_5\cos^{n-5}\theta\sin^5\theta-\cdots \end{cases}$$

（証明）
$(a+bi)^n = {}_nC_0a^n+{}_nC_1a^{n-1}(bi)+{}_nC_2a^{n-2}(bi)^2+{}_nC_3a^{n-3}(bi)^3+\cdots$
$(a-bi)^n = {}_nC_0a^n-{}_nC_1a^{n-1}(bi)+{}_nC_2a^{n-2}(bi)^2-{}_nC_3a^{n-3}(bi)^3+\cdots$
ゆえに、
$$\frac{(a+bi)^n+(a-bi)^n}{2}$$

$= {}_nC_0a^n + {}_nC_2a^{n-2}(bi)^2 + {}_nC_4a^{n-4}(bi)^4 + {}_nC_6a^{n-6}(bi)^6 + \cdots$
$= {}_nC_0a^n - {}_nC_2a^{n-2}b^2 + {}_nC_4a^{n-4}b^4 - {}_nC_6a^{n-6}b^6 + \cdots$
$\dfrac{(a+bi)^n - (a-bi)^n}{2i}$
$= {}_nC_1a^{n-1}b + {}_nC_3a^{n-3}b^3i^2 + {}_nC_5a^{n-5}b^5i^4 + {}_nC_7a^{n-7}b^7i^6 + \cdots$
$= {}_nC_1a^{n-1}b - {}_nC_3a^{n-3}b^3 + {}_nC_5a^{n-5}b^5 - {}_nC_7a^{n-7}b^7 + \cdots$
ゆえに、

$$\begin{cases} \cos n\theta = {}_nC_0\cos^n\theta - {}_nC_2\cos^{n-2}\theta\sin^2\theta + {}_nC_4\cos^{n-4}\theta\sin^4\theta - \cdots \\ \sin n\theta = {}_nC_1\cos^{n-1}\theta\sin\theta - {}_nC_3\cos^{n-3}\theta\sin^3\theta + {}_nC_5\cos^{n-5}\theta\sin^5\theta - \cdots \end{cases}$$

3.11 $\sin n\theta$

$\sin 2\theta$
$= {}_2C_1\cos\theta\sin\theta$
$= 2\cos\theta\sin\theta$

$\sin 3\theta$
$= {}_3C_1\cos^2\theta\sin\theta - {}_3C_3\sin^3\theta$
$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$
$= -4\sin^3\theta + 3\sin\theta$

$\sin 4\theta$
$= {}_4C_1\cos^3\theta\sin\theta - {}_4C_3\cos\theta\sin^3\theta$
$= 4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)$
$= 4\sin\theta\cos\theta(1 - 2\sin^2\theta)$

$\sin 5\theta$
$= {}_5C_1\cos^4\theta\sin\theta - {}_5C_3\cos^2\theta\sin^3\theta + {}_5C_5\sin^5\theta$
$= 5\sin\theta(1-2\sin^2\theta+\sin^4\theta) - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$
$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$
$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

$\sin 6\theta$
$= {}_6C_1\cos^5\theta\sin\theta - {}_6C_3\cos^3\theta\sin^3\theta + {}_6C_5\cos\theta\sin^5\theta$
$= 2\sin\theta\cos\theta(3\cos^4\theta - 10\cos^2\theta\sin^2\theta + 3\sin^4\theta)$
$= 2\sin\theta\cos\theta(3(1-2\sin^2\theta+\sin^4\theta) - 10(1-\sin^2\theta)\sin^2\theta + 3\sin^4\theta)$
$= 2\sin\theta\cos\theta(3 - 6\sin^2\theta + 3\sin^4\theta - 10\sin^2\theta + 10\sin^4\theta + 3\sin^4\theta)$
$= 2\sin\theta\cos\theta(16\sin^4\theta - 16\sin^2\theta + 3)$

$\sin 7\theta$
$= {}_7C_1\cos^6\theta\sin\theta - {}_7C_3\cos^4\theta\sin^3\theta + {}_7C_5\cos^2\theta\sin^5\theta - {}_7C_7\sin^7\theta$
$= 7\sin\theta(1-3\sin^2\theta+3\sin^4\theta-\sin^6\theta) - 35\sin^3\theta(1-2\sin^2\theta+\sin^4\theta) + 21\sin^5\theta(1-\sin^2\theta) - \sin^7\theta$
$= 7\sin\theta - 21\sin^3\theta + 21\sin^5\theta - 7\sin^7\theta - 35\sin^3\theta + 70\sin^5\theta - 35\sin^7\theta + 21\sin^5\theta - 21\sin^7\theta - \sin^7\theta$

$= -64\sin^7\theta + 112\sin^5\theta - 56\sin^3\theta + 7\sin\theta$

3.12 $\cos n\theta$

$\cos 2\theta$
$= {}_2C_0\cos^2\theta - {}_2C_2\sin^2\theta$
$= \cos^2\theta - \sin^2\theta$

$\cos 3\theta$
$= {}_3C_0\cos^3\theta - {}_3C_2\cos\theta\sin^2\theta$
$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$
$= 4\cos^3\theta - 3\cos\theta$

$\cos 4\theta$
$= {}_4C_0\cos^4\theta - {}_4C_2\cos^2\theta\sin^2\theta + {}_4C_4\sin^4\theta$
$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + 1 - 2\cos^2\theta + \cos^4\theta$
$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$
$= 8\cos^4\theta - 8\cos^2\theta + 1$

$\cos 5\theta$
$= {}_5C_0\cos^5\theta - {}_5C_2\cos^3\theta\sin^2\theta + {}_5C_4\cos\theta\sin^4\theta$
$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-2\cos^2\theta+\cos^4\theta)$
$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$
$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

$\cos 6\theta$
$= {}_6C_0\cos^6\theta - {}_6C_2\cos^4\theta\sin^2\theta + {}_6C_4\cos^2\theta\sin^4\theta - {}_6C_6\sin^6\theta$
$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-2\cos^2\theta+\cos^4\theta) - (1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta)$
$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta - 30\cos^4\theta + 15\cos^6\theta - 1 + 3\cos^2\theta - 3\cos^4\theta + \cos^6\theta$
$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$

$\cos 7\theta$
$= {}_7C_0\cos^7\theta - {}_7C_2\cos^5\theta\sin^2\theta + {}_7C_4\cos^3\theta\sin^4\theta - {}_7C_6\cos\theta\sin^6\theta$
$= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1-2\cos^2\theta+\cos^4\theta) - 7\cos\theta(1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta)$
$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta - 70\cos^5\theta + 35\cos^7\theta - 7\cos\theta + 21\cos^3\theta - 21\cos^5\theta +$
$7\cos^7\theta$
$= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$